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Magnetovac solutions of the static Einstein-Maxwell equations

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Abstract : Magnetostatic solutions of Einstein-Maxwell field equations for a mass possessing a magnetic dipole moment are constructed by the method developed by Gutsunaev and Manko [*Gen. Rel. Grav.* **20** 327 (1988), *Phys. Lett.* **A132** 85 (1988)]. The generated solutions are well behaved at spatial infinity. However, in absence of magnetic field, the solutions do not reduce to the Schwarzschild metric.

Keywords : General relativity, Einstein-Maxwell equation, magnetovac solutions

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1. Introduction

Magnetosatic solutions of Einstein-Maxwell field equations are of much interest for their possible applications in astrophysics. In the past many research workers presented a number of static magnetovac solutions in General Relativty [1–6] to describe the exterior gravitational field of a mass endowed with a magnetic dipole moment. However, all those solutions generated so far do not always reduce to the Schwarzschild static limit in absence of magnetic field. Different transformation techniques, such as the transformation technique of Bonnor [1,2], the method of Gutsunaev-Manko (GM) [3,4], the technique of Das-Chaudhuri [6] *etc* were developed to generate new static magnetovac solutions.

In this paper, we construct a static magnetovac solution of Einstein-Maxwell field equations using the method of Gutsunaev-Manko [3,4]. The magnetic dipole moment of the source is evaluated. It is found that the solutions are asymptotically flat. However, they do not reduce to the Schwarzschild form in absence of magnetic field.

In Section 2, the formalism of Gutsunaev-Manko is described in brief. The constructed solutions are presented in Section 3. Our conclusion follows in Section 4.

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2. Gutsunaev-Manko formalism

Consider a static axially symmetric metric in Weyl coordinates (r,z),

$$ds^2 = f^{-1}\left[e^{2\gamma}\left(dr^2 + dz^2\right) + r^2 d\phi^2\right] - f dt^2, \tag{1}$$

where f and γ are functions of r, z only. The Einstein-Maxwell magnetostatic field equations are :

$$\nabla^2 f = f^{-1}(\nabla f)^2 + 2r^{-2}f^2(\nabla A_3)^2, \tag{2}$$

$$\nabla(r^{-2} f \nabla A_3) = 0, \tag{3}$$

$$4\gamma_{,r} = rf^{-2}\left(f_{,r}^2 - f_{,z}^2\right) + 4r^{-1}f\left(A_{3,r}^2 - A_{3,z}^2\right), \tag{4}$$

$$4\gamma_{,z} = rf^{-2} f_{,r} f_{,z} + 4r^{-1} f A_{3,r} A_{3,z}, \tag{5}$$

where a comma denotes partial differentiation and

$$\nabla^2 = \partial_{rr} + r^{-1}\partial_r + \partial_{zz},$$

$$\nabla^2 \equiv \hat{r}_0 \partial_r + \hat{z}_0 \partial_z, \tag{6}$$

$\hat{r}_0$ and $\hat{z}_0$ are unit vectors,

A_3 is the magnetic component of electromagnetic four-potential and is related to the pseudomagnetic potential A'_3 by [4],

$$A'_{3,r} = r^{-1} f A_{3,z},$$
$$A'_{3,z} = r^{-1} f A_{3,r}, \tag{7}$$

We now define two new real potentials

$$\varepsilon_1 = f^{1/2} + A'_3,$$
$$\varepsilon_2 = f^{1/2} - A'_3. \tag{8}$$

According to GM, the field eqs. (2)–(5) may be written in terms of ε_1 and ε_2 as [4] :

$$(\varepsilon_1 + \varepsilon_2)\nabla^2\varepsilon_1 = 2(\nabla\varepsilon_1)^2, \tag{9}$$

$$(\varepsilon_1 + \varepsilon_2)\nabla^2\varepsilon_2 = 2(\nabla\varepsilon_2)^2, \tag{10}$$

$$\gamma_{,r} = 4r(\varepsilon_1 + \varepsilon_2)^{-2}(\varepsilon_{1,r}\varepsilon_{2,r} - \varepsilon_{1,r}\varepsilon_{2,z}), \tag{11}$$

$$\gamma_{,z} = 4r(\varepsilon_1 + \varepsilon_2)^{-2}(\varepsilon_{1,r}\varepsilon_{2,z} - \varepsilon_{1,z}\varepsilon_{2,r}), \tag{12}$$

Gutsunaev and Manko showed that if ψ be any Laplace's solution *i.e.* ψ satisfies the equation

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2} = 0, \tag{13}$$

then the potentials ε_1 and ε_2 are expressed in the form [4],

$$\varepsilon_1 = e^{\psi}\left[1 - \frac{2(1-a)(1-b)}{x(1-ab) + y(b-a) + (1-a)(1-b)}\right], \tag{14}$$

$$\varepsilon_2 = e^{\psi}\left[1 - \frac{2(1+a)(1+b)}{x(1-ab) - y(b-a) + (1+a)(1+b)}\right]. \tag{15}$$

The parameters a and b satisfy the following first order differential equations, given in prolate spheroidal coordinates (x,y),

$$a_{,x} = a(x-y)^{-1}\left[(xy-1)\psi_{,x} + (1-y^2)\psi_{,y}\right],$$

$$a_{,y} = a(x-y)^{-1}\left[-(xy-1)\psi_{,x} + (xy-y^2)\psi_{,y}\right],$$

$$b_{,x} = -b(x+y)^{-1}\left[(xy-1)\psi_{,x} + (1-y^2)\psi_{,y}\right],$$

$$b_{,x} = -b(x+y)^{-1}\left[-(x^2-1)\psi_{,x} + (xy+1)\psi_{,y}\right]. \tag{16}$$

Prolate spheroidal coordinates (x,y) are related to Weyl coordinates (r,z) by the relations :

$$r^2 = K^2(x^2-1)(1-y^2),$$
$$z = Kxy, \tag{17}$$

where K is a real constant.

With the help of eqs. (8),(14) and (15), the metric potentials f and A'_3 are given by

$$f = e^{2\psi}\left[\frac{C^2x^2 - E^2y^2 - D^2 + F^2}{(Cx+D)^2 - (Ey-F)^2}\right] \tag{18}$$

$$A'_3 = e^{2\psi}\,\frac{2(CFx + DEy)}{(Cx+D)^2 - (Ey-F)^2}, \tag{19}$$

where

$$C = 1 - ab,$$
$$D = 1 + ab,$$
$$E = b - a,$$
$$F = b + a, \tag{20}$$

a and b can be determined from eqs. (16).

The formulation for obtaining static magnetovac solutions is thus complete. The mass and the magnetic dipole moment of the source can be evaluated from eqs. (18) and (19) respectively.

3. The solutions

We take the seed Laplace's solution ψ, in prolate spheroidal coordinates (x,y) as

$$\psi = q(x+y)^{-1}, \tag{21}$$

where q is a constant.

Using eqs. (16) and (21), the functions a and b are found to be

$$a = -\alpha \exp[qy(x+y)^{-1}],$$
$$b = -\alpha \exp[-q(1+xy)(x+y)^{-2}], \tag{22}$$

where α is another constant.

From eqs. (18)–(22), the metric potentials f and A'_3 are obtained as

$$f = \left(\frac{N_2}{N_1} \right) \exp\left[2q(x+y)^{-1}\right], \tag{23}$$

$$A_3' = \left(\frac{N_3}{N_1} \right) \exp\left[q(x+y)^{-1}\right], \tag{24}$$

where N_1, N_2 and N_3 are given by

$$N_1 = \left[(x+1) - \alpha^2 (x-1) e^{K_3}\right]^2,$$

$$-\alpha^2 \left[(1+x)e^{K_1} - (1-y)e^{K_2}\right]^2 \tag{25}$$

$$N_2 = \left(1 + \alpha^2 e^{K_3}\right)^2 x^2 - \alpha^2 \left(e^{K_2} + e^{K_1}\right)^2 y^2$$

$$+ \alpha^2 \left(e^{K_2} - e^{K_1}\right)^2 - \left(1 - \alpha^2 e^{K_3}\right)^2, \tag{26}$$

$$N_3 = \alpha \left(e^{K_2} - e^{K_1}\right)^2 \left(1 + \alpha^2 e^{K_3}\right) x$$

$$+ \alpha \left(e^{K_2} + e^{K_1}\right)\left(1 - \alpha^2 e^{K_3}\right) y, \tag{27}$$

and

$$K_1 = qy(x + y)^{-1},$$
$$K_2 = -q(1 + xy)(x + y)^{-2},$$
$$K_3 = -q(1 - y^2)(x + y)^{-2}. \tag{28}$$

The asymptotic expressions for f and A'_3 are given by

$$f = 1 + \frac{2\left\{q\left(1+\alpha^2\right) - 2\left(1-\alpha^2\right)\right\}}{\left(1+\alpha^2\right)} \frac{1}{x}.$$

$$\frac{4\left(1-\alpha^2\right)^2 \left\{1 - q\left(\frac{1+\alpha^2}{1-\alpha^2}\right)\right\} + q\left(1+\alpha^2\right)(q-y)}{\left(1+\alpha^2\right)^2} \frac{1}{x^2} + \ldots \tag{29}$$

$$A_3' = \frac{4\alpha}{\left(1+\alpha^2\right)^2} \cdot \frac{\left[\left(1-\alpha^2\right) - q\left(1+\alpha^2\right)\right] y}{x^2} + \ldots \tag{30}$$

Under the coordinate transformations

$$Kx = R - m, \quad y = \cos\theta, \quad m \text{ is a constant,} \tag{31}$$

eqs. (29) and (30) take the form :

$$f = 1 + \frac{2K}{\left(1+\alpha^2\right)} \left[q\left(1+\alpha^2\right) - 2\left(1-\alpha^2\right)\right] \frac{1}{R}$$

$$+ \frac{2K}{\left(1+\alpha^2\right)^2} \left[CK + m\left(1+\alpha^2\right)\right.$$

$$\left.\left\{q\left(1+\alpha^2\right) - 2\left(1-\alpha^2\right)\right\}\right] \frac{1}{R^2} + \ldots \tag{32}$$

$$A_3' = \frac{4\alpha K^2}{\left(1+\alpha^2\right)^2} \cdot \frac{\left[\left(1-\alpha^2\right) - q\left(1+\alpha^2\right)\right] \cos\theta}{R^2} \tag{33}$$

where

$$C = 4\left(1-\alpha^2\right) \frac{1+\alpha^2}{1-\alpha^2} + q\left(1+\alpha^2\right)^2 (q - \cos\theta).$$

The mass (M) and the dipole moment (μ) of the source, obtained from eqs. (32) and (33), are given by

$$M = K\left[2\left(1-\alpha^2\right) - q\left(1+\alpha^2\right)\right]\left(1+\alpha^2\right)^{-1}, \tag{34}$$

$$\mu = 4\alpha K^2 \left[\left(1-\alpha^2\right) - q\left(1+\alpha^2\right)\right]\left(1+\alpha^2\right)^{-2} \tag{35}$$

It is to be noted from eqs. (32)–(33) that as $R \to \infty$, $f \to 1$ and $A'_3 \to 0$ *i.e.* the solutions are asymptotically well-behaved. The parameter α may be identified with the magnetic parameter. When $\alpha = 0$, the dipole moment μ vanishes and f becomes

$$f = \left(\frac{(x-1)}{(x+1)} \right) \exp\left[2q(x+y)^{-1}\right] \tag{36}$$

The solution (23) thus does not reduce to the Schwarzschild form in absence of magnetic field. This is because of our choice of the seed metric. If the seed solution is properly chosen then one can get Schwarzschild solution from the magnetovac metric obtained by this method.

4. Conclusion

The magnetostatic solutions of a source in vacuum endowed with a magnetic dipole moment are presented in the paper. The mass and the dipole moment of the source are evaluated. The solutions are found to be spatially well behaved at infinity containing monople, dipole and other higher mass multipole moments. However, in absence of magnetic field (*i.e.* $\alpha = 0$) the solutions do not reduce to the Schwarzschild metric.

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